

The Parabola as an Equilibrium Plasma-Vacuum Interface

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The free-boundary MHD equilibrium is considered in the case of plane symmetry and constant current profile. We present a global solution of the problem, given in terms of elementary functions, which shows a parabolic plasma-vacuum interface.

1. Formulation of the Problem

When ideal conductivity is assumed, the magneto-hydrostatic equations read

$$\nabla p = \mathbf{j} \times \mathbf{B}, \quad \mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0 \quad (1)$$

with p denoting the pressure, \mathbf{j} the current density and \mathbf{B} the magnetic field strength. These equations may be simplified by the ansatz

$$\mathbf{B} = \nabla \Psi \times \nabla z + B_z \nabla z \quad (2)$$

with an arbitrary flux function $\Psi(x, y)$ and an arbitrary axial field component $B_z(x, y)$. The current density is then given by

$$\mathbf{j} = \frac{1}{\mu_0} (-\Delta \Psi \nabla z + \nabla B_z \times \nabla z) \quad (3)$$

and the pressure balance equation becomes

$$\nabla(\mu_0 p + \frac{1}{2} B_z^2) + \Delta \Psi \nabla \Psi - (\nabla B_z \times \nabla \Psi \cdot \nabla z) \nabla z = 0. \quad (4)$$

Equation (4) implies that p and B_z are constant along the level lines of Ψ , which are just the magnetic field lines. This means that, if these lines are connected, p and B_z may be expressed as functions of Ψ and (4) is equivalent to the Lüst-Schlüter-Grad-Shafranov equation [1–3]

$$\Delta \Psi + \mu_0 j_z(\Psi) = 0, \quad (5)$$

$$j_z(\Psi) = \frac{d}{d\Psi} \left(p(\Psi) + \frac{1}{2\mu_0} B_z^2(\Psi) \right). \quad (6)$$

The restriction to plane symmetry implies a considerable simplification because the – 2-dimensional – Laplace operator is now involved in (5), which makes

potential theory and conformal mapping techniques available for solving the problem (note that in general the Stokes operator Δ^* would arise!).

To simplify the problem further, we choose j_z to be constant in the plasma region R_p :

$$j_z(\Psi) = j_0 \quad \text{in } R_p, \quad (7)$$

whereas the vacuum region R_v is characterized by vanishing current density. The solution Ψ is normalized in such a way that the plasma-vacuum interface Γ is given by the level line $\Psi = 0$. In order for the magnetic field to be continuous across the interface, it is also necessary that $\partial \Psi / \partial n$ be continuous across Γ .

Finally, we want to cast our problem in dimensionless form, using r_0 , j_0 and $\mu_0 j_0 r_0^2$ as units for r , j , and Ψ .

The free-boundary problem can then be summarized as follows:

$$\Delta \Psi_p + 1 = 0 \quad \text{in } R_p, \quad (8)$$

$$\Psi_p = 0 \quad \text{on } \Gamma, \quad (9)$$

$$\Delta \Psi_v = 0 \quad \text{in } R_v, \quad (10)$$

$$\Psi_v = 0 \quad \text{on } \Gamma, \quad (11)$$

$$\frac{\partial \Psi_p}{\partial n} = \frac{\partial \Psi_v}{\partial n} \quad \text{on } \Gamma. \quad (12)$$

For this kind of problem very few results are available. In the case where the domain $\Omega := R_p \cup R_v$ is bounded and Ψ is assumed to be constant on $\partial \Omega$, it has been proved that a solution always exists for a linear current profile in [4, 5] and for more general profiles in [6]. For $\Omega = \mathbb{R}^2$ no similar general theorems are known, and also exact solutions are scarcely available. As was pointed out in [7], the general case Ψ_v shows point- or line-singularities corresponding to (in physical terms) wires or current sheets in the vacuum region. This means solutions existing in the entire plane are rare

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exceptions. Moreover, simple exact solutions often serve as test pieces for complex equilibrium codes.

Reference [8] gives a simple exact solution with an elliptic interface and a quadrupole field as asymptotic vacuum solution. Here we present a solution with unbounded plasma region R_p exhibiting a parabolic interface and different asymptotics in R_p and R_v .

2. Solution

The general strategy is as follows: A solution of the plasma problem (8), (9) can easily be found for simple boundary curves Γ like the parabola. To solve the vacuum problem (10), (11), we first look for a conformal mapping which maps R_v in a region \tilde{R}_v with an even simpler boundary $\tilde{\Gamma}$ (unit circle or straight line) and which shows no singularities (such as branching points or cuts) in \tilde{R}_v . Next we try a harmonic series as an ansatz for $\tilde{\Psi}_v$. Condition (11) can now easily be satisfied at the simplified boundary $\tilde{\Gamma}$; the crucial point, however, is to satisfy also the matching condition (12).

Let us apply this method in the parabola case. The plasma solution is

$$\Psi_p = \frac{1}{2} (4p^2(p^2 - x) - y^2) \quad (13)$$

and vanishes at the parabola

$$y^2 = 4p^2(p^2 - x), \quad (14)$$

where p is a real positive parameter.

As conformal mapping (see, for example, [9, 10]) we choose the square root

$$w = f(z) = \sqrt{z}, \quad (15)$$

which maps the z -plane, cut along the negative real axis, on the right w -half-plane

$$u = \sqrt{\frac{1}{2}(r+x)}, \quad v = \operatorname{sgn}(y) \sqrt{\frac{1}{2}(r-x)}, \quad r^2 = x^2 + y^2. \quad (16)$$

Moreover, f maps the parabola (14) on the straight line

$$u = p \quad (17)$$

and the outer region of the parabola to the right of $u = p$ (see Fig. 1).

In \tilde{R}_v we try the complex function ansatz

$$\begin{aligned} \tilde{\Psi}_v(u, v) &= \operatorname{Re} F(w), \\ F(w) &= \sum_{n=1}^{\infty} a_n (w - w_0)^n, \end{aligned} \quad (18)$$

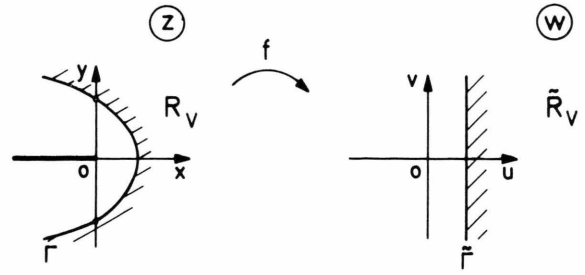


Fig. 1.

which satisfies the boundary condition (11) at $\tilde{\Gamma}$ for $w_0 = p$ and a_{2n-1} real and a_{2n} imaginary.

Because Ψ is constant along Γ , the matching condition (12) is – up to a sign – equivalent to

$$|\nabla \Psi_p|^2 = |\nabla \Psi_v|^2 \quad \text{on } \Gamma. \quad (19)$$

The right-hand side of (19) may be expressed by the complex derivative

$$\frac{dF}{dz} = \frac{\partial F}{\partial x} = \frac{\partial \Psi}{\partial x} - i \frac{\partial \Psi}{\partial y} \quad (20)$$

or

$$\left| \frac{dF}{dz} \right|^2 = |\nabla \Psi_v|^2. \quad (21)$$

When evaluated on $\tilde{\Gamma}$, (19) takes the form

$$|\nabla \tilde{\Psi}_p(x(u, v), y(u, v))|^2 \cdot \left| \frac{dz}{dw} \right|^2 \Big|_{u=p} = \left| \frac{dF}{dw} \right|^2 \Big|_{u=p}. \quad (22)$$

Using (13) and (15) in the left-hand-side of (22) yields

$$|\nabla \tilde{\Psi}_p|^2 \cdot \left| \frac{dz}{dw} \right|^2 \Big|_{u=p} = 16p^2 v^4 + 32p^4 v^2 + 16p^6. \quad (23)$$

When the ansatz (18) is inserted in the right-hand side of (22), a comparison with (23) shows that at most the first three coefficients in (18) are different from zero. With $a_2 = -ib_2$, b_2 real we get

$$\begin{aligned} \left| \frac{dF}{dw} \right|^2 \Big|_{u=p} &= a_1^2 + 4a_1 b_2 v + (4b_2 - 6a_1 a_3) v^2 \\ &\quad - 12b_2 a_3 v^3 + 9a_3^2 v^4. \end{aligned} \quad (24)$$

Comparison with (23) determines the coefficients a_1 , b_2 , a_3 :

$$\begin{aligned} a_1 &= \pm 4p^3, \\ b_2 &= 0, \\ a_3 &= \mp 4/3 p. \end{aligned} \quad (25)$$

To fix the sign we check the matching condition (12) at the special point $z = p^2 \sim w = p$:

$$\begin{aligned}\frac{\partial \Psi_p}{\partial x} &= -2p^2, \\ \frac{\partial \Psi_v}{\partial x} &= Re \frac{dF}{dw} \cdot \frac{dw}{dz} = \pm 2p^2,\end{aligned}\quad (26)$$

and so the upper sign has to be discarded. Altogether the vacuum solution Ψ_v can be written in the complex form

$$\begin{aligned}\Psi_v(x, y) &= Re F(w(z)), \\ F(z) &= -4p^3(\sqrt{z-p}) + \frac{4}{3}p(\sqrt{z-p})^3.\end{aligned}\quad (27)$$

3. Discussion

The solution (13), (27) is symmetric with respect to the x -axis and shows a simple stagnation point in $z = 4p^2$. “Simple” means that the level lines running through that point intersect at right angles.

The asymptotic behaviour in R_v can be read off most simply in polar coordinates (r, ϕ) :

$$\Psi_v \sim Re z^{3/2} = r^{3/2} \cos \frac{3}{2} \phi, \quad (28)$$

and so Ψ_v has the lines

$$y = \pm \sqrt{3}x \quad (29)$$

as asymptotics.

If the magnetic field is plane ($B_z = 0$), (6) allows a statement about the pressure gradient at the interface. It points in the same direction as the gradient of Ψ , i.e. into the plasma (see (26)). So the solution (13), (27) has the property of confining the plasma.

Interesting on its own is the limit case $p \rightarrow 0$. In this case the plasma region degenerates to the negative real axis. In order to keep the total current finite, we change the normalization of the current density to

$$\hat{j} = -a/p \quad (30)$$

with an arbitrary real constant $a \neq 0$. The current density integrated in the y -direction now remains finite in the limit $p \rightarrow 0$,

$$\hat{j} = -a/p \int_{-2p\sqrt{p^2-x}}^{2p\sqrt{p^2-x}} 1 dy = -4a\sqrt{p^2-x} \rightarrow -4a\sqrt{-x}, \quad (31)$$

and corresponds to the (x -dependent) line density,

$$\hat{j}(x) = -4a\sqrt{-x} \delta(y). \quad (32)$$

The vacuum solution takes the form

$$\tilde{\Psi}_v(x, y) \rightarrow 4/3 a Re z^{3/2} = 4/3 a r^{3/2} \cos \frac{3}{2} \phi, \quad (33)$$

the stagnation point has moved to the (degenerated) interface, and the separatrix now consists of the straight zero level lines, intersecting at angles of $2\pi/3$.

In conclusion, two points should be noted:

1. The mapping (15) is not the only one possible. Another one would be

$$w = \tilde{f}(z) = 2\sqrt{\frac{p^2}{z}} - 1, \quad (34)$$

which maps the outer region of the parabola on the interior of the unit circle. In that case an ansatz for the complex function F of the form

$$\tilde{F}(w) \sim \sum_n f_n^\pm \left(w^n \pm \frac{1}{w^n} \right) + g(\log w) \quad (35)$$

is successful and leads to the already known vacuum solution (27). The last point is not accidental, because the problem of analytic continuation of the plasma solution into the vacuum region is unique according to the theorem of Cauchy-Kowalevski (see, for example, [11]).

2. The reader may ask whether other conic sections for which similarly simple plasma solutions exist are solutions of the plasma problem (8)–(12), too. The answer is yes and no. The ellipse is indeed a solution, as was shown a long time ago by using elliptic coordinates [8]. On the other hand, in the case of the hyperbola the problem of continuation inevitably runs into a singularity; there is thus no global solution (at least in the case where the plasma solution is a second-order polynomial).

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